Simultaneously stabilizing networked systems with minimal communication

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***Abstract*— Self-triggered feedback control is used as a means to reduce the energy and communication requirements of networked systems. Triggered-control schemes result in aperiodic communications, with a potential for communication conflicts when multiple feedback loops are closed over a shared network. In this paper, we analyze the necessary/sufficient conditions for simultaneous stabilizability of a set of nonlinear systems over a network from the perspective of scheduling theory, using Lyapunov functions and input to state stability. We then propose a recursively feasible self-triggering scheme that minimizes the usage of the communication channel while ensuring the stability of all systems.**

***Index Terms*— Self-triggered control, networked systems, scheduling, simultaneous stabilizability**

I. Introduction

Aperiodic feedback control, and particularly event- or self-triggered feedback control, are typically used as a means to reduce the energy and communication requirements of a control system, minimizing the number of updates of the control variable per unit time [1]–[4]. Advantages become evident in the context of control loops closed over a communication network, where a parsimonious usage of the communication channel allows the control loop to coexist with other services sharing the same communication medium. In this case, the framework of self-triggered control is particularly interesting since it avoids all communications when inputs are not being updated.

Such designs raise a number of issues, besides the obvious challenge of enforcing the stability of the closed-loop system despite the reduced communication rate. Control over a network requires the encoding of measures and control signals in digital form, bringing about the limitations of a quantized representation of real-valued data that led to the modern theory of quantized control [5]–[8]. Furthermore, self-triggered control of multiple nodes involves the coupling between a problem of communication scheduling and one of ensuring or even optimizing control performances given the chosen communication schedule. This is a well-known issue in real-time systems and network control systems literature [9]–[12], discussed at least since [13]. Joint scheduling and control design was discussed for almost as long [14]–[19]. In all these cases, the focus is typically on ensuring the performance of a given feedback loop closed over a network, possibly coping with the presence of other tasks competing for the same communication resource (as was the case in [2], one of the first publications introducing the current framework for event-triggered control). The implications of parallel control loops competing for the same communication resource, and their joint design, are less frequently investigated. An exception is in the research line that started with [20] (see e.g., [21]–[23]) where constraints on wide classes of communication protocols, in the form of a Maximum Allowable Transfer Interval (MATI), were computed to ensure the stability of multiple feedback loops closed over a network. Our work here is close in spirit to some of the ideas that stemmed from this literature. However, here we address the more general problem of determining whether a protocol exists at all, which will stabilize the network of nonlinear nodes. Once the stabilizability of the network is established, we then address the problem of developing a dynamic communication protocol that also minimizes the usage of the communication medium. In this sense, our target is closest to [24], which provides a theoretical basis for schedulability analysis and scheduler synthesis for LTI systems. In this paper, we address roughly this same class of problems, but with a few relevant differences. We introduce in Section II the notion of *connection patterns*, which allows us to describe a wide variety of network structures. Our main contributions are then in Sections III and IV. In Section III we derive the conditions for the simultaneous stabilizability of a set of systems over a network, and we discuss the relations between the simultaneous stabilizability of nonlinear systems and of their linearization. Then, in Section IV we define a centralized triggering law that retains the main property of self-triggered isolated systems: the connection of one or more systems is triggered only when necessary to ensure asymptotic stability. By construction, the centralized triggering function does not introduce any communication overhead since the state measurements are communicated by each system *only* at the time when its connection is triggered.

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II. Problem Setting

A. Definition of the model class

Let us consider a discrete-time nonlinear system

$$x(t+1) = f(x(t), u(t)), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, \tag{1}$$

with an equilibrium at $x = u = 0$ and with f continuously differentiable in x and u. We assume throughout the paper that the equilibrium of (1) is not asymptotically stable, since the other case is not of interest in this work. Consider then a feedback control

$$u(t) = k(x(t)) + e(t), \tag{2}$$

continuously differentiable in x and such that $k(0) = 0$, which makes the equilibrium $x = 0, u = 0$ of the closed-loop system

$$x(t+1) = f\left(x(t), k(x(t)) + e(t)\right)$$

input-to-state stable (ISS) with respect to errors e in the evaluation of the feedback $k(x)$. Assume, furthermore, that the Jacobian of the closed-loop equilibrium, $\frac{\partial}{\partial x} f\left(x, k(x)\right)|_{x=0}$, is Hurwitz. In the following sections, we will work with the above nonlinear system, and with its linearization at the origin. To distinguish the two, we denote by M^{nl} the nonlinear system with dynamics (1), (2), and by M^{l} its linearization at the origin.

Let us then define

$$x(1, \xi, u) := f(\xi, u)$$

the state reached by M^{nl} from initial condition ξ with input u in one step, and let $x(t, \xi, \cdot)$ be the iteration of $x(1, \xi, \cdot)$ for t times. According to a well-known result (see e.g., [25], or [26] for the equivalent in continuous-time systems), the origin of the closed-loop system is ISS for the input $e(t)$ if and only if it admits a Lyapunov function $V(x)$, such that

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|), \, \forall\, x \in \mathbb{R}^n \tag{3}$$

and

$$V(x(1, \xi, k(\xi) + e)) - V(\xi) \le \\ -\alpha_3(|\xi|) + \gamma_e(|e|), \quad \forall\, \xi \in \mathbb{R}^n, \forall\, e \in \mathbb{R}^n \tag{4}$$

where α_1, α_2, α_3, are $\mathcal{K}_\infty$ functions and γ_e is a $\mathcal{K}$ function.

Throughout the paper we take the following assumption.

Assumption 1: V is quadratic.

Restricting our choice of Lyapunov functions to the family of quadratic functions allows us to ensure that a Lyapunov function for a nonlinear system M^{nl} is a Lyapunov function for its linearization M^{l} as well. This will be used in Section III to prove the necessary conditions on the stabilizability of a network of systems M^{nl}. However notice that other than Lemma 3, Theorem 6, and Theorem 7, which regard these necessary conditions, all results in this paper hold independently of Assumption 1.

Consider now the case where the input u is only updated on a subsequence $\{t_1, t_2, \ldots\} \subset \mathbb{Z}$ of time instants, and assume, for the moment, that the only source of error e is the discrepancy between $k(x(t))$ and $k(x(t_j))$, $t \in [t_j, t_{j+1}]$. The equilibrium remains closed-loop asymptotically stable provided that the sequence of update times is constructed such that

$$\gamma_e(|k(x(t)) - k(x(t_j))|) \le (1-\sigma)\alpha_3(|x(t)|), \tag{5}$$

for some $\sigma \in (0, 1)$.

This is the idea behind the event and self-triggered control definitions in [27] and, in continuous time, in [2], [28], [29]. At a generic time t, with $t_j \le t \le t_{j+1}$, an event-triggered control law is simply synthesized by triggering an update if

$$\gamma_e\big(\,|k(x(t+1)) - k(x(t_j))|\,\big) > (1-\sigma)\alpha_3\big(|x(t)|\big).$$

The above condition implies that negativity of the left-hand side of (4) is no longer ensured unless the input is updated at time t. Note that, while the above condition is formulated in event form, the arguments $x(t+1)$ and $x(t_j)$ are fully determined at time t_j, since the model (1) is not affected by external noise. Therefore, the triggering process is effectively self-driven.

Consider, however, the case where M^{nl} is but one of several nodes in a networked control system, and the update of the input u of a node must compete for resources with the input updates of other nodes. While the problem of scheduling input updates in the presence of other preemptive processes was already discussed (see e.g., [2]), in this case, preemption of one update in favor of another may result in violation of constraint (5) for the preempted node. Design of the update sequences for all network nodes must, in this case, confront the additional constraint of ensuring the asymptotic stability of all nodes at the same time. In order to properly discuss the implications of this fact, we need to introduce some terms and definitions.

B. Definition of the network topology and communication constraints

In the sequel, we will frequently refer to intervals of integers. To keep our notation short, we will denote by $\mathbb{I}_a^b$ the set of integers $\{a, a+1, \ldots, b\}$.

Consider a network of q nodes competing for a communication resource. Assume that each node is modeled as in (1), with possibly different systems for different nodes, and assume the presence of a centralized scheduler that can assign the resource based on knowledge of the state of all nodes. When a node does not receive an update on its input, it uses the last received input.

In the simplest scenario, we may imagine that only one node of the network can access the resource at each time instant: we will call this the *trivial connection topology*. In this case, we could formulate the problem by simply asking that (5) be satisfied without updating more than one input at a time. In more general cases, however, connections may follow more elaborate patterns: information regarding multiple input values could be packed in a single packet or transmitted during a single time slot, or multiple communication channels could be used. Also, in the case where the input is updated based on information gathered by one out of multiple antennas, satellites, or cameras that collectively cover the whole network, the sets of nodes covered by different antennas, satellites, or cameras could be of different cardinality.

To encompass all these scenarios under a common framework, we introduce the concept of *connection pattern*. Consider a set $\boldsymbol{\mathcal{C}}$ of tuples C where

$$C := (c_1, \ldots, c_m). \tag{6}$$

Different tuples may have a different number of elements (different m), and the elements of a tuple C encode the nodes

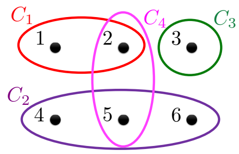

Fig. 1. A network with 6 nodes and 4 connection patterns. We encode with $\mathcal{C} := \{(1,2),(4,5,6),(3),(2,5)\}$ the condition that the input of nodes within the same ellipse can be updated simultaneously.

whose input is updated under the corresponding connection pattern (if $c \in C$, then node c is connected when connection pattern C is chosen, see Fig. 1 for an example). Set $\mathcal{C}$ thus defines the connection topology, by describing all possible combinations of nodes whose inputs can be updated simultaneously. Assuming, as we did above, to have a network of q nodes, in the case of a trivial connection topology the set $\mathcal{C}$ contains exactly q tuples, each made of a single symbol. In the case of a multi-channel communication network, when up to m arbitrary inputs can be updated at the same time, the set $\mathcal{C}$ contains $\sum_{i=1}^{m} \binom{q}{i}$ tuples, corresponding to all possible combinations of up to m symbols out of the q available ones. Of course, more complex constraints can be encoded by suitably shaping the elements in $\mathcal{C}$.

Stabilization of multiple feedback loops over a network is typically discussed in terms of *communication protocols* and Lyapunov-based stability properties of the couple protocol-system [21]. In our case, results are more easily illustrated in terms of the *schedule*, that is, the sequence of connection patterns that ensue from a given protocol. Schedules will therefore be the central ingredient in our line of attack. Scheduling connection pattern C at time t means that only nodes $j \in C$ can communicate at time t. A schedule S is, then, an infinite sequence of connection patterns:

$$S := (C_1, C_2, \ldots).$$

C. Problems Statement

We can now formulate the two problems object of this paper. We shall first investigate the simultaneous stabilizability of the network.

Problem 1: Given a network of nodes M_j^{nl}, $j \in \mathbb{I}_1^q$, and the constraints induced by a set of connection patterns $\mathcal{C}$, determine under what conditions it is possible to ensure the asymptotic stability of all the nodes in the network.

Then, given a network whose nodes can be simultaneously stabilized, we will address the design of a self-triggered control scheme.

Problem 2: Given a network of nodes M_j^{nl}, $j \in \mathbb{I}_1^q$, and the constraints induced by a set of connection patterns $\mathcal{C}$, devise a self-triggered control scheme to ensure asymptotic stability of all the nodes in the network.

We address Problem 1 in Section III and Problem 2 in Section IV.

III. Simultaneous stabilizability

It is immediately apparent that Problem 1 is tightly related to a scheduling problem: does a schedule S exist that ensures the asymptotic stability of all nodes? We clarify this connection first, in Section III-A, by characterizing bounds on the frequency with which a single node needs input updates to guarantee stability, and then, in Section III-B, by using these constraints to formulate simultaneous stabilizability conditions. The main results are in Theorems 5 and 7, which provide separate sufficient and necessary conditions.

A. p-step stabilizability of a single node

We begin by defining a condition on the maximum number of time steps that a node can remain disconnected, without becoming unstable. The condition will be applied both to nonlinear systems (M^{nl}) and to their linearizations (M^{l}), hence we write it in terms of a generic system M.

Definition 1 (p-step stabilizable pair): Consider a system M with dynamics (1) and (2), and a Lyapunov function $V(x)$ satisfying (3) and (4). The pair (M, V) is p-step stabilizable with parameter $\sigma \in (0,1)$ if

$$\gamma_e(|k(x(t,\xi,k(\xi))) - k(\xi)|) \le (1-\sigma)\alpha_3(|x(t,\xi,k(\xi))|), \quad \forall\, \xi \in \mathbb{R}^n,\ \forall\, t \in \mathbb{I}_0^{p-1}. \tag{7}$$

Clearly, the above definition is only meaningful if $p \ge 1$, so we will always assume it in the following. The definition states that a pair (M, V) is p-step stabilizable for parameter σ if all p induced systems that are iterations of the original system, with input u held constant for 1 up to p steps, are simultaneously stabilized by the feedback $u = k(x)$, since (7) with (4) implies that the Lyapunov function V decreases at the rate upper bounded by $-\sigma\alpha_3(x)$, for all p systems:

$$V(x(t+1,\xi,k(\xi))) - V(x(t,\xi,k(\xi))) \le -\sigma\alpha_3(|x(t,\xi,k(\xi))|), \quad \forall\, \xi \in \mathbb{R}^n,\ \forall\, t \in \mathbb{I}_0^{p-1}. \tag{8}$$

One may therefore see p as a MATI for the discrete-time system (1), (2), under the family of protocols that updates u at arbitrary intervals between 1 and p steps. Note that, in general, a pair (M, V) that is p-step stabilizable for a given σ may not be so for a larger σ, and a system M that is p-step stabilizable paired with V may not be so when paired with a different Lyapunov function.

We are therefore implicitly assuming that the control design for the isolated node (and therefore the choice of the Lyapunov function and the tolerable convergence speed) is completed before the stabilizability properties of the node in the network are discussed.

Later we will also use the more stringent definition of *maximally p-step stabilizable pair for parameter* σ, meaning a pair that is p-step stabilizable but not $(p+1)$-step stabilizable, keeping parameter σ fixed. The results that follow in this Section provide a bound for the p-step stabilizability of a linear system, and a relation between these bounds and those of a nonlinear system of which this is the linearization. The proofs are reported in the Appendix.

Let us consider the linear system M^{l} defined as

$$x(t+1) = Ax(t) + Bu(t), \quad u(t) = Kx(t) + e(t), \tag{9}$$

such that matrix $A+BK$ is asymptotically stable. The closed-loop system

$$x(t+1) = (A+BK)x(t) + Be(t)$$

has a Lyapunov function $V = x^\top P x$ such that

$$(A+BK)^\top P(A+BK) - P = -Q, \tag{10}$$

with P and Q positive definite.

Lemma 1: The linear system M^{l} and the Lyapunov function satisfying (10) form a maximally p-step stabilizable pair for parameter $\sigma \in (0,1)$, with

$$p := \{\max\ \bar{p} : D(t) \preceq -\sigma G^\top(t) Q G(t),\ \forall t \in \mathbb{I}_0^{\bar{p}-1}\},$$

where $G(t) := \left(A^t + \sum_{j=0}^{t-1} A^j BK\right)$ and $D(t) := (AG(t) + BK)^\top P (AG(t) + BK) - G(t)^\top P G(t)$.

Obviously, decreasing the value of σ in the above equation relaxes the constraint on p, at the expense of a slower convergence rate of the closed-loop system.

Notice that, while the above lemma ensures the existence of a finite upper bound to the step stabilizability parameter of a linear system, we can prove the existence of such an upper bound also for the nonlinear system M^{nl}.

Lemma 2: For any $\sigma^{\mathrm{nl}} \in (0,1)$ there exists a finite p^{nl} such that the pair (M^{nl}, V) is maximally p^{nl}-step stabilizable with parameter σ^{nl}.

Furthermore, under Assumption 1, we can relate the step stabilizability properties of a nonlinear system M^{nl} and its linearization M^{l} as follows.

Lemma 3: Under Assumption 1, if the pair (M^{nl}, V) is p^{nl}-step stabilizable with parameter σ^{nl}, then the pair (M^{l}, V) is p^{l}-step stabilizable with parameter σ^{l}, with $0 < \sigma^{\mathrm{l}} \leq \sigma^{\mathrm{nl}}$ and $p^{\mathrm{l}} \geq p^{\mathrm{nl}}$.

Example 1: Take as M^{nl} the following nonlinear system

$$x(t+1) = \arctan(x(t)) + \frac{x(t)}{2} + u(t),$$
$$u(t) = -0.51 \arctan(x(t)) + e(t),$$

and take $V(x) := x^2$ and $\alpha_3(|x|) := \frac{x^2}{100}$. If we set $x(0) = \xi$ and assume that $u(t)$ is only updated at $t = 0$, the closed loop dynamics become

$$x(t+1) = \arctan(x(t)) + \frac{x(t)}{2} - 0.51 \arctan(\xi).$$

The linearization M^{l} of the above system at $x(t) = 0$ and $\xi = 0$ is equal to

$$x(t+1) = 1.5x(t) - 0.51\xi,$$

which satisfies (8) with $p := p^{\mathrm{l}} = 10$ for sufficiently small σ, as shown in Figure 2. The pair (M^{l}, V) is 10-step stabilizable, for sufficiently small σ. Consider however the full nonlinear model M^{nl}. By explicitly evaluating the largest p such that

$$V(x(t,\xi,k(\xi))) - V(x(t-1,\xi,k(\xi))) < 0,\ \forall t \leq p,$$

as a function of ξ, we obtain the curve in Figure 3. As expected, near the origin and for small enough σ, (8) is satisfied for $p = 10$. However, there are larger initial conditions for which (8) is satisfied only up to $p := p^{\mathrm{nl}} = 6$, no matter how small parameter σ is taken. The pair (M^{nl}, V) is, therefore, at most 6-step stabilizable, even for extremely small σ.

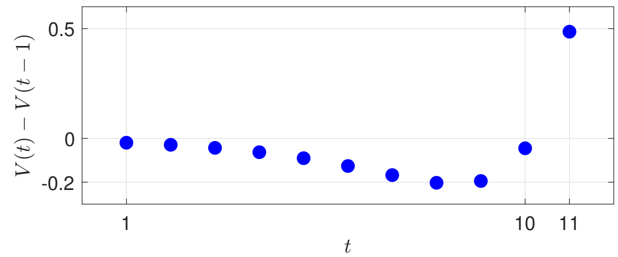

Fig. 2. Variation in V, with $\xi = \sigma = 1$, for M^{l} in Example 1. Note how V decreases up to $t = 10$.

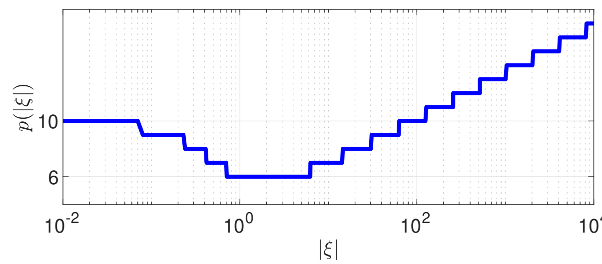

Fig. 3. The maximum value of p for which (8) is satisfied in system M^{nl} in Example 1, for an arbitrarily small parameter $\sigma > 0$, as a function of $|\xi|$.

B. Simultaneous stabilizability

We have thus established a relation between the step stabilizability of a nonlinear system M^{nl} and its linearization M^{l}, and we have provided, in Lemma 1, a formula to compute parameter p^{l}. We will see shortly how this parameter can be used to formulate conditions for the simultaneous stabilizability of all nodes in the network. We can now address Problem 1.

We start by formalizing the objective of controlling all systems to the origin while enforcing the convergence properties guaranteed by condition (5) as the following *simultaneous stabilizability* property, for a set of q systems M_j^{nl}, paired with as many quadratic Lyapunov functions V_j.

Definition 2 (Simultaneous stabilizability, SS): A set of systems $M_j^{\mathrm{nl}}, j \in \mathbb{I}_1^q$, is SS if there exist parameters σ_j and a schedule $S := (C_1, C_2, \ldots)$, $C_k \in \boldsymbol{\mathcal{C}}$, that enforces (5) for all systems.

To familiarize ourselves with the above definition, let us consider a set of nodes with trivial connection topology. In this case, each symbol C in the schedule defines the connection of a single node c_j, so that we can identify the connection pattern C with the symbol c_j. If we assume that each pair (M_j^{nl}, V_j) is p_j-step stabilizable for some finite p_j, then SS is guaranteed provided we find a schedule $S := (c_{j_1}, c_{j_2}, \ldots)$ such that each symbol c_j appears at least once every p_j steps. This is an example of a well-known scheduling problem, known as the *pinwheel scheduling* [30]. The pinwheel scheduling problem is NP-complete, but numerous results exist that allow one to decide the existence of a schedule for large families of pinwheel problems with low computational effort [30]–[32]. In particular, quite interestingly for the scope of this paper, it was proved in [30] that all pinwheel problems admit a schedule if and only if they admit a periodic schedule, with period bounded by a function of the parameters p_j. This means that the schedulability of an instance of the pinwheel problem can always be decided by searching over a finite set of periodic schedules. This may lead us to think that proving SS can be done using standard scheduling machinery. This is, unfortunately, only partially true, for two reasons. First, the general case of nontrivial connection topology brings some theoretical complications. Second, parameters p_j are only upper bounds to the time that a node can remain disconnected. In the following, we introduce a set of sufficient and set of necessary conditions for SS. We assume that all pairs (M_j^{nl}, V_j) (resp. (M_j^{l}, V_j)) are maximally p_j^{nl} (resp. p_j^{l})-step stabilizable for arbitrarily small $\sigma_j > 0$, and we know from Lemma 2 that p_j^{nl} and p_j^{l} are always finite.

Let us begin by addressing the sufficient conditions. Our strategy is to approximate a problem with arbitrary connection topology with one with trivial connection topology, and then to use standard sufficient conditions from the pinwheel scheduling theory.

Given a set of q nodes with r (nontrivial) connection patterns, consider the vector $\mathbf{p} := (p_1, \ldots, p_q) \in \mathbb{R}_+^q$ of step stabilizability constraints, and the following mixed-integer optimization problem

$$\min_{\rho_i, \eta_{i,j}} \quad \sum_{i=1}^{r} \rho_i \tag{11a}$$

$$\text{s.t.} \quad \rho_i \geq \frac{1}{p_j}\eta_{i,j}, \qquad \forall\, i \in \mathbb{I}_1^r, j \in \mathbb{I}_1^q, \tag{11b}$$

$$\sum_{i: j \in C_i} \eta_{i,j} = 1, \qquad \forall\, j \in \mathbb{I}_1^q, \tag{11c}$$

$$\eta_{i,j} \in \{0, 1\}, \qquad \forall\, i \in \mathbb{I}_1^r, \forall\, j \in \mathbb{I}_1^q. \tag{11d}$$

In the above problem, index i ranges over the available connection patterns, while index j ranges over the nodes. We can therefore see the variables $\eta_{i,j}$ as the elements of a matrix η with columns corresponding to nodes, and rows to connection patterns. Constraint (11c) imposes that columns of η have a single nonzero element, which defines an assignment of the node to a unique connection pattern. Constraint (11b) then states that for each connection pattern C_i, parameter ρ_i be larger than the inverse of the step-stabilizability constraint p_j for all nodes j assigned to connection pattern C_i. Parameter ρ_i is thus akin to a frequency with which connection pattern C_i should be scheduled, in order to ensure the stability of all the nodes that were assigned to it. Call $\rho^*(\mathbf{p}) \in \mathbb{R}_+^r, \eta^*(\mathbf{p}) \in \{0,1\}^{r \times q}$ an optimal solution of the above problem. By (11a) and (11b), for each connection pattern C_i, either $\rho_i = 0$ (no node was assigned to C_i), or there exists at least one node j such that $\rho_i^*(\mathbf{p}) = 1/p_j$. We may consider these as the *critical* nodes, as they constrain the frequency with which connection patterns should be scheduled. We may then select one critical node for each connection pattern, and prove that SS of this set of nodes with trivial connection topology implies SS of the original set of nods with the original connection topology. We formalize this in Lemma 4, using a *selection function*, which returns the set of indices of the selected systems, and a connection pattern *translation function*, which given the index j of one of the selected systems returns the connection pattern to which the system was assigned.

Definition 3 (Selection function $\Sigma(\mathbf{p}, \boldsymbol{\mathcal{C}})$): Given $\mathbf{p}$ and $\boldsymbol{\mathcal{C}}$, return a subset $\{j\}$ of nodes such that, for each $i \in \mathbb{I}_1^r$ with $\rho_i^*(\mathbf{p}) > 0$, there exists a unique $j \in \Sigma(\mathbf{p}, \boldsymbol{\mathcal{C}})$ such that $j \in C_i$, $\eta_{i,j}^*(\mathbf{p}) = 1$, $\rho_i^*(\mathbf{p}) = \frac{1}{p_j}$.

Note that the function above is required to return one symbol $j \in \mathbb{I}_1^q$ satisfying the stated constraints, but the solution of (11) may admit multiple such j. Which symbol is chosen is irrelevant to the results that follow.

Definition 4 (Translation function $\Theta(\mathbf{p}, \boldsymbol{\mathcal{C}}, j)$): Given $\mathbf{p}$, $\boldsymbol{\mathcal{C}}$, and a symbol $j \in \Sigma(\mathbf{p}, \boldsymbol{\mathcal{C}})$, return the unique C_i such that $\eta_{i,j}^* = 1$; i.e., $\Theta(\mathbf{p}, \boldsymbol{\mathcal{C}}, j) := \{C_i \text{ with } \eta_{i,j}^* = 1\}$.

For better readability, in the following, we write the selection function $\Sigma(\mathbb{I}_1^q)$ and the translation function $\Theta(j)$, leaving implicit the problem parameters $(\mathbf{p}, \boldsymbol{\mathcal{C}})$ while explicating the effect of the selection function of selecting a subset $\Sigma(\mathbb{I}_1^q)$ of the symbols $\mathbb{I}_1^q$.

Consider a set $\mathbb{I}_1^q$ of nodes (1) endowed with Lyapunov functions to form p_i-step stabilizable pairs, and a set $\boldsymbol{\mathcal{C}} := \{C_1, \ldots, C_r\}$ with nontrivial connection topology.

Lemma 4: A sufficient condition for simultaneous stabilization of nodes $\mathbb{I}_1^q$ is that the subnet with nodes $\Sigma(\mathbb{I}_1^q)$ be SS with trivial connection topology $\hat{\boldsymbol{\mathcal{C}}} := \Sigma(\mathbb{I}_1^q)$. Moreover, if a schedule $\hat{S} = (\hat{C}_1, \hat{C}_2, \ldots)$ in the trivial connection topology $\hat{\boldsymbol{\mathcal{C}}}$ stabilizes the subnet $\Sigma(\mathbb{I}_1^q)$, then the schedule $S := (C_1, C_2, \ldots)$ with $C_i := \Theta(\hat{C}_i)$ simultaneously stabilizes all nodes $\mathbb{I}_1^q$.

The above lemma, whose proof is in the Appendix, provides a means to prove SS for a network with nontrivial connection topology, by checking SS for a subnetwork with trivial connection topology, and a means to translate a stabilizing schedule for the subnetwork into one for the full network. The computation of a stabilizing schedule for the subnetwork (with trivial connection topology) can be performed using the pinwheel scheduling algorithms mentioned in [30]–[32], giving us the following result.

Theorem 5 (SS, sufficient conditions): Sufficient conditions for SS of nodes $\mathbb{I}_1^q$ are

$$\sum_{i=1}^{r} \rho_i^*(\mathbf{p}^{\text{nl}}) \leq \frac{3}{4}, \tag{12}$$

or

$$\sum_{i=1}^{r} \rho_i^*(\mathbf{p}^{\text{nl}}) \leq \frac{5}{6} \text{ and } \exists\, i \in \mathbb{I}_1^r : \ \rho_i^*(\mathbf{p}^{\text{nl}}) = \frac{1}{2}. \tag{13}$$

Proof: By Lemma 4, if the nodes $\Sigma(\mathbb{I}_1^q)$ (defined by solving (11) and with the corresponding trivial connection topology) are SS, then the nodes $\mathbb{I}_1^q$ with nontrivial connection topology are SS. A sufficient condition for the SS of nodes $\Sigma(\mathbb{I}_1^q)$ is the existence of a schedule where each node $j \in \Sigma(\mathbb{I}_1^q)$ is connected at least once every p_j^{nl} steps. Conditions (12) and (13) are then proved to be sufficient for the existence of such a schedule in [33]. ■

Example 2: Consider again the network in Fig. 1, and assume that, for a given parameter σ, all nodes are maximally p^{nl}-step stabilizable with $p_1^{\text{nl}} = 10$, $p_2^{\text{nl}} = 2$, $p_3^{\text{nl}} = 8$, $p_4^{\text{nl}} = 10$, $p_5^{\text{nl}} = 3$, $p_6^{\text{nl}} = 9$. A naïve attempt might be to try to use only connection patterns C_1, C_2, C_3, which together cover all the network nodes, ignoring C_4. In this case, we easily notice that the schedule $S := (C_1, C_2, C_1, C_2, \ldots)$ is the only possible schedule that connects nodes 2 and 5 frequently enough, but since it never connects node 3 it will not simultaneously stabilize the network. However, the solution of (11) is $\rho^*(\mathbf{p}^{\text{nl}}) = (1/10, 1/9, 1/8, 1/2)$, and

$$\eta^*(\mathbf{p}^{\text{nl}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have that $\sum_{i=1}^{4} \rho_i^*(\mathbf{p}^{\text{nl}}) \simeq 0.836 < \frac{5}{6}$, $\rho_4^*(\mathbf{p}) = 1/2$, so by Theorem 5 the network is SS. To find a stabilizing schedule, we note that the selection function has value $\Sigma(\mathbb{I}_1^6) = (1, 2, 3, 6)$. Nodes $(1, 2, 3, 6)$ with trivial connection topology are simultaneously stabilized, for example, by the schedule $\hat{S} := (2, 1, 2, 3, 2, 6, 2, 1, 2, 3, 2, 6, \ldots)$. We have $\Theta(1) = C_1$, $\Theta(2) = C_4$, $\Theta(3) = C_3$, $\Theta(6) = C_2$, therefore by Lemma 4 the full network is simultaneously stabilized by the schedule $S =: (C_4, C_1, C_4, C_2, C_4, C_3, C_4, C_1, C_4, C_2, C_4, C_3, \ldots)$.

Let us now attack the problem of deriving the necessary conditions for SS. We begin with the following result.

Theorem 6: A necessary condition for the SS of $M_j^{\text{nl}}, j \in \mathbb{I}_1^q$, is that there exists a periodic schedule

$$S := (C_1, C_2, \ldots, C_l, C_1, C_2, \ldots),$$

of period $l \leq \prod_j p_j^{\text{l}}$, such that every symbol j belongs to C_i for at least one C_i in every subsequence of p_j^{l} successive connection patterns.

Proof: Assume that the systems M_j^{nl} are SS by some schedule $S := (C_1, C_2, \ldots)$, possibly non-periodic. This implies, by Lemma 3, that the same schedule simultaneously stabilizes M_j^{l} at the origin: the linearizations and the corresponding Lyapunov functions form p_j^{l}-step stabilizable pairs, for suitable $\sigma_j > 0$, with $p_j^{\text{l}} \geq p_j^{\text{nl}}$. Define an infinite sequence of q-dimensional vectors $(D_1, D_2, \ldots)$, so that there is a vector D_k for each $t_k \geq 0$, and that $D_{k,j}$ (the j-th element of vector D_k) represents the count, at time t_k, of the number of steps since the input of system j was last updated. If S simultaneously stabilizes the set of linearized systems, given that p_j^{l} is maximal for any $\sigma_j > 0$, it must be that $0 \leq D_{k,j} < p_j^{\text{l}}$, therefore vectors D_k can assume at most $\prod_{j=1}^{q} p_j^{\text{l}}$ different values. This means that the sequence $(D_1, D_2, \ldots)$ must have two identical vectors, at most $\prod_{j=1}^{q} p_j^{\text{l}}$ steps apart. Let k_1, k_2 denote the time instants corresponding to these two identical vectors; we have $k_2 - k_1 \leq \prod_{j=1}^{q} p_j^{\text{l}}$. Let $S_{\text{period}} := (C_{k_1}, \ldots C_{k_2 - 1})$ be the corresponding finite subsequence of S. The periodic schedule obtained by iterating S_{period} indefinitely ensures that every symbol $j \in C_i$ for at least one C_i in every subsequence of p_j^{l} successive connection patterns, and therefore simultaneously stabilizes the set of linearized systems. ■

Part of the proof structure for Theorem 6 is taken from Theorem 2.1 in [30] which, however, addresses a simpler setup of the pinwheel scheduling of a set of q symbols. The above theorem constrains the SS of an arbitrary network with the SS of the corresponding network of linearized systems by a periodic schedule. We can obtain a necessary condition for the existence of such as schedule by recurring to a modified version of problem (11). Consider the problem

$$\min_{\hat{\rho}_i, \hat{\eta}_{i,j}} \quad \sum_{i=1}^{r} \hat{\rho}_i \tag{14a}$$

$$\text{s.t.} \quad \sum_{i=1}^{r} \hat{\rho}_i \hat{\eta}_{i,j} \geq \frac{1}{p_j}, \qquad \forall\, i \in \mathbb{I}_1^r, j \in \mathbb{I}_1^q, \tag{14b}$$

$$\hat{\eta}_{i,j} = 1, \qquad \forall\, i \in \mathbb{I}_1^r, j \in C_i, \tag{14c}$$

$$\hat{\eta}_{i,j} = 0, \qquad \forall\, i \in \mathbb{I}_1^r, j \notin C_i \tag{14d}$$

and let $\hat{\rho}^*(\mathbf{p}) \in \mathbb{R}_+^r, \hat{\eta}^*(\mathbf{p}) \in \{0, 1\}^{r \times q}$ be an optimal solution. Notice that (14c) and (14d) are stating that $\hat{\eta}_{i,j} = 1$ if and only if $j \in C_i$. In other words, variables $\eta_{i,j}$ have a fixed value, so that, contrary to (11), the above is a Linear Program (LP). We leave them in the problem statement to highlight the formal symmetry between problems (11) and (14). Indeed, while in (11) parameter ρ_i represents the frequency with which a connection pattern should be scheduled to ensure, by itself, the stability of all nodes that were assigned to it, in (14) a node j is simultaneously assigned by (14c) and (14d) to all connection patterns C_i with $j \in C_i$. Then, interpreting $\hat{\rho}_i$ as the average portion of time slots allocated to connection pattern C_i, (14b) is stating that the set of connection patterns C_i that connect node j should collectively be allocated at least $1/p_j$ of the time slots. Using the optimal solution to the above LP, and Theorem 6, we have the following.

Theorem 7 (SS, necessary condition): A necessary condition for SS of nodes $\mathbb{I}_1^q$ is that

$$\sum_{i=1}^{r} \hat{\rho}_i^*(p^{\text{l}}) \leq 1, \tag{15}$$

Proof: According to Theorem 6, SS implies the existence of a periodic schedule S such that every symbol j belongs to a C_i for at least one C_i in every subsequence of p_j^{l} successive connection patterns. Consider a single period of the schedule of length L. Every connection pattern C_i appears $\hat{\rho}_i L$ times in a period, for some $\hat{\rho}_i \in [0, 1]$, and

$$\sum_{i=1}^{r} L \hat{\rho}_i = L. \tag{16}$$

Then, SS implies that in schedule C_i, connection patterns that contain symbol j appear in at least a fraction L/p_j of all the steps of a period L. Remembering that, according to (14c) and (14d), $\hat{\eta}_{i,j} = 1$ if and only if $j \in C_i$, this with (16) imply that

$$\sum_{i=1}^{r} L \hat{\rho}_i \eta_{i,j} \geq \frac{L}{p_j^{\text{l}}}, \; \forall\, j \in \mathbb{I}_1^q.$$

The above inequality implies (14b), and therefore guarantees the existence of a feasible solution $\hat{\rho}(\mathbf{p}^{\text{l}})$ to (14). Furthermore, by (16) such a solution satisfies $\sum_i \hat{\rho}_i(\mathbf{p}^{\text{l}}) = 1$, and this implies condition (15). ■

Theorems 5 and 7 partially solve Problem 1 by providing separate necessary and sufficient conditions for SS. As far as we know, a unified set of necessary and sufficient conditions for SS cannot be given, except for the case where all nodes are linear.

Corollary 8: For a network of linear systems, Theorem 6 provides necessary and sufficient conditions

Proof: If all nodes are linear, then $p_j^{\rm l}$ and $p_j^{\rm nl}$ coincide. Then the conditions of Theorem 6 define a schedule that simultaneously stabilizes all nodes. They are therefore sufficient, and not only necessary. ■

An implication of the above corollary is that with linear nodes a stabilizing schedule exists if and only if a periodic stabilizing schedule exists, while with nonlinear nodes we can expect that any stabilizing schedule settles on a periodic sequence, once the state is close enough to the equilibrium.

Note that the necessary condition (15) is based on $\hat{\rho}^*(\mathbf{p})$, i.e., the optimizer of (14). On the other hand, the sufficient conditions (12) and (13) are based on $\rho^*(\mathbf{p})$, i.e., the optimizer of (11). Computation of the two, therefore, requires the solution of distinct optimization problems. Furthermore, there exist networks that are SS, and hence satisfy the necessary condition, while not satisfying the sufficient conditions, and there exist networks that satisfy the necessary condition while not being SS, as the next two examples show.

Example 3: Consider a set of 3 linear systems, described as in (1), with the trivial connection topology and $p_1^{\rm l} = p_2^{\rm l} = 3,\ p_3^{\rm l} = 4$. In this case, $\hat{\rho}_1^*(\mathbf{p}^{\rm l}) = \hat{\rho}_2^*(\mathbf{p}^{\rm l}) = \frac{1}{3}$ and $\hat{\rho}_3^*(\mathbf{p}^{\rm l}) = \frac{1}{4}$. Therefore, $\sum_i \hat{\rho}_i^*(\mathbf{p}^{\rm l}) = \frac{11}{12}$ which satisfies (15). Since the systems are linear, $\mathbf{p}^{\rm nl} = \mathbf{p}^{\rm l}$, therefore $\sum_i \rho_i^*(\mathbf{p}^{\rm nl}) = \frac{11}{12}$. As a result, the sufficient conditions (12) and (13) are not satisfied. Nevertheless, the simple periodic schedule $S = (1, 2, 3, \dots)$ simultaneously stabilizes the set of systems. This example shows that the sufficient conditions (12) and (13) are not necessary.

Example 4: Consider Example 3 with $p_1^{\rm l} = 2,\ p_2^{\rm l} = 3,\ p_3^{\rm l} = 12$ and the trivial connection topology. Similar to Example 3, $\sum_i \hat{\rho}_i^*(\mathbf{p}^{\rm l}) = \sum_i \rho_i^*(\mathbf{p}^{\rm nl}) = \frac{11}{12}$, which implies the necessary condition (15) is satisfied but the sufficient conditions (12) and (13) are not. Contrary to Example 3, one can show by exhaustive search that the systems are not SS. This example shows that the necessary condition (15) is not sufficient.

To conclude, let us explore the application of Theorems 5 and 7 to the nonlinear system of Example 1.

Example 5: Consider a network with q nodes with the nonlinear dynamics of Example 1. In case of trivial connection topology we have $p^{\rm l} = 10$ and $p^{\rm nl} = 6$, equal for all nodes. Sufficient condition (12) ensures that a network of at least 4 nodes can be simultaneously stabilized, while necessary condition (15) limits the largest SS network to 10 nodes. In this example, one can easily verify that a Round Robin schedule will simultaneously stabilize up to 6 nodes.

Now consider the non-trivial set of connection patterns $\boldsymbol{\mathcal{C}} = \{1,\ (2,3),\ (4,5,6), \dots,\ (\dots, q-1, q)\}$. In this case, the sufficient condition (15) holds for $q \le 55$; however, the necessary condition (12) only holds for $q \le 10$.

IV. Centralized self-triggered scheduling

Consider now a set of nodes and corresponding Lyapunov functions, so that the nodes are SS by a periodic schedule, each node and Lyapunov function forming a p_i-step stabilizable pair for some parameter σ_i. This means that there exists at least one periodic schedule S of connection patterns stabilizing all systems. In the case of linear nodes, this is not a restrictive assumption, as observed in Corollary 8. Call $\mathbf{R}$ any set of finite sequences of connection patterns that form minimal periods of periodic stabilizing schedules, that is, sequences of connection patterns that, iterated, form the full schedule, and that cannot be decomposed into identical subsequences. In the following, with a little abuse of notation, we identify by S both the (infinite) periodic schedule and the subsequence that forms a minimal period, so that $S \in \mathbf{R}$ identifies the set of infinite schedules whose minimal periods are sequences in $\mathbf{R}$. Assume also that the set $\mathbf{R}$ includes all rotations of each sequence, i.e., if $(C_1, \dots, C_L) \in \mathbf{R}$ then $(C_k, \dots, C_L, C_1, \dots, C_{k-1}) \in \mathbf{R}$, $\forall\, k \in \mathbb{I}_1^L$. For each node j in the network, define $\underline{t}_j$ as the last time node j was connected and $\underline{\xi}_j$ as the corresponding state. Define

$$\begin{aligned} p_{j,\xi}(t) := \max_{p\in\mathbb{N}} \{ p : \gamma_{j,e}(|k_j(x_j(t-\underline{t}_j, \underline{\xi}_j, k_j(\underline{\xi}_j))) - k_j(\underline{\xi}_j)|) \\ \le (1-\sigma_j)\alpha_{j,3}(|x_j(t-\underline{t}_j, \underline{\xi}_j, k_j(\underline{\xi}_j))|)\ \forall\, t \in \mathbb{I}_0^{p-1} \} \quad (17) \end{aligned}$$

Differently than in (7), in the above equation, the inequality is required to hold only for a specific value of $\underline{\xi}_j$. Note that, by construction, $p_{j,\xi}(t+1) = p_{j,\xi}(t) - 1$.

Then, call $\bar{t}_j(S, t)$ the next time node j will be connected, if schedule S is used from time t onward.

$$\bar{t}_j(S, t) := t + \left(\min_{\tau \in \mathbb{I}_1^L} \tau : j \in S(\tau) \right).$$

Note that $\bar{t}_j(S, t)$ is guaranteed to be upper bounded by $t + L$, given that S is periodic of period L. We can now formulate the following centralized triggering function, exploiting the fact that it is safe to leave the node disconnected for $\bar{t}_j(S, t) - t$ time steps and knowing that it has already been disconnected for $p_{j,\xi}(t)$ time steps. The triggering function can then be defined as

$$T(t) := \max_{S\in\mathbf{R}} \min_j \left(p_{j,\xi}(t) - (\bar{t}_j(S,t) - t) \right), \quad (18)$$

and the corresponding triggered event

$$E(t) : \mathbb{N} \to \boldsymbol{\mathcal{C}}, \quad E(t) := S^*(1),$$

where S^* is the maximizing schedule in (18) and $S^*(1)$ is the first connection pattern in the schedule. Function $T(t)$ measures the deadline before which a node will need to be connected, to enforce stability, in the best possible schedule among those available in $\mathbf{R}$.

Theorem 9 (Solution to Problem 2): Assume that the nodes are SS by a periodic schedule and that $T(t) > 0$ at time $t = 0$. Then, Problem 2 is solved by connecting nodes with connection pattern $E(t)$ when $T(t+1) \le 0$, or leaving all nodes disconnected otherwise.

Proof: We need to show that, with the suggested connection strategy,

$$\begin{aligned} &\gamma_{j,e}(|k(x_j(t-\underline{t}_j, \underline{\xi}_j, k_i(\underline{\xi}_j))) - k(\underline{\xi}_j)|) \\ &\qquad \le (1-\sigma_j)\alpha_{3,j}(|x_i(t-\underline{t}_j, \underline{\xi}_j, k_i(\underline{\xi}_j))|),\ \forall\, t \ge 0, \quad (19) \end{aligned}$$

since this implies, through (4), that

$$
\begin{aligned}
&V(x_j(t-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j)))-V(x_j(t-1-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j)))\\
&\qquad\le -\sigma\alpha_{j,3}(|x_i(t-1-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j))|),\ \forall\,\xi_j\in X_j.
\end{aligned}
$$

Once this is proven, the result follows from the Lyapunov theorem.

In the rest of the proof, we use the symbol S^* to denote the maximizing schedule in (18) computed at time t, leaving its dependence on t implicit. We start by noting that the assumption of simultaneous stabilizability by a periodic schedule implies that $\mathbf{R}$ is nonempty, therefore S^* exists. The condition $T(0)>0$ implies that at time 0, $\bar{t}_j(S^*,0)<p_{j,x}$, for all j, i.e., each node is scheduled to be connected before its own $p_{j,x}$. By the definition of $p_{j,\xi}$ this means that

$$
\begin{aligned}
&\gamma_e(|k_j(x_j(t-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j)))-k_j(\underline{\xi}_j)|)\\
&\qquad\qquad\le(1-\sigma)\alpha_{3,j}(|x_j(t)|),\ \forall\,t\in\mathbb{I}_0^{\bar{t}_j(S^*)}.
\end{aligned}
$$

From then on, feasibility of the schedule S^* implies that each node j is connected at least once every p_j steps, ensuring

$$
\begin{aligned}
&\gamma_e(|k_j(x_j(t-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j)))-k_j(\underline{\xi}_j)|)\\
&\qquad\qquad\le(1-\sigma)\alpha_3(|x_j(t)|),\ \forall\,t\ge\bar{t}_j(S^*).
\end{aligned}
$$

Condition (19) follows from the two above inequalities. ■

Next, we provide an example that uses Theorem 9 to minimize communication while guaranteeing SS.

Example 6: Consider a network of 4 nonlinear nodes with

$$
\begin{aligned}
x_j(t+1)=&\begin{bmatrix}1 & (0.12-0.02\cdot j)\\ 1 & 0\end{bmatrix}x_j(t)\\
&+\begin{bmatrix}0\\1\end{bmatrix}u_j(t)+\begin{bmatrix}0\\-0.05\sin(x_{j,1}(t))\end{bmatrix},
\end{aligned}
$$

where

$$
u_j(t)=\begin{bmatrix}-0.1 & -0.1\end{bmatrix}x_j(t),
$$

and assume the trivial connection topology. Consider $\sigma_j=0.2$ for all j and $V_j(x)=x_j^\top P_j x_j$, where P_j is the solution to the Lyapunov equation for the system linearized at the origin, with Q_j equal to the identity matrix. Furthermore, let $\alpha_{j,3}(|x_j|)=x_j^\top Q_j x_j$.

We can show through Lemma 1 that the linearizations of the given systems and Lyapunov functions are maximally p_j-step stabilizable with $(p_1,p_2,p_3,p_4)=(4,6,7,8)$. The systems and Lyapunov function pairs also satisfy Assumption 1 such that, by Lemma 3, these are also upper bounds to the p-step stabilizability of the nonlinear systems. A grid search over the 2-dimensional state space suggests that these bounds are tight. Since $\sum_i\rho_i^*(\mathbf{p}^{\mathrm{nl}})=\frac{1}{4}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}<\frac{3}{4}$, the sufficient condition (12) in Theorem 5 is satisfied and a periodic stabilizing schedule exists. In this case in particular a stabilizing schedule is obtained by repeating sequence $(1,2,3,4)$. The results of a simulation in which a transmission is triggered as devised in Theorem 9, letting

$$
\mathbf{R}:=\{(1,2,3,4),\ (2,3,4,1),\ (3,4,1,2),\ (4,1,2,3)\}
$$

is shown in Figure 4. The top panel shows the values of the Lyapunov functions over time, after setting $x_j(0)^\top=[2\ 1]$

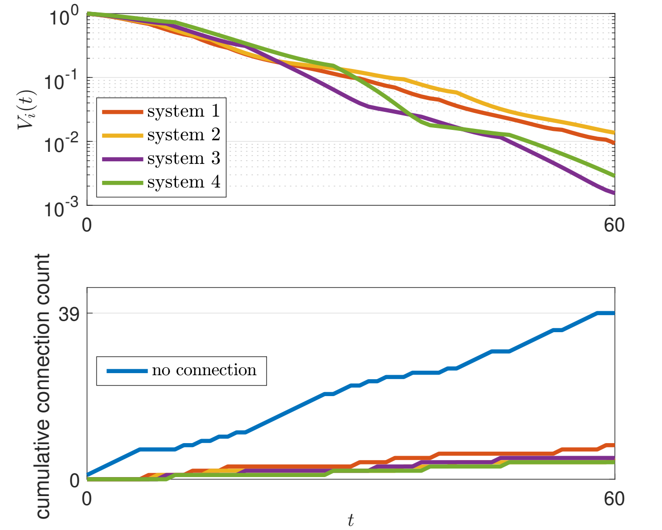

Fig. 4. The top panel shows the normalized value of the Lyapunov functions for each of the 4 systems in Example 6, as a function of time. The bottom panel shows the cumulative number of connections of each of the systems; the blue line is the cumulative number of time instants when the communication channel is left unused.

and normalizing $V_j(t)$ with respect to $V_j(0)$ for all $j\in\mathbb{I}_1^4$. The Lyapunov functions are strictly decreasing, as expected. Using the inequalities

$$
V(t+1)-V(t)\le-\sigma x^\top(t)x(t),
$$

and

$$
V(t)\le\lambda_{\max}(P)
$$

where $\lambda_{\max}$ is the dominant eigenvalue of P, one obtains

$$
V(t)\le V(0)\left(1-\frac{\sigma}{\lambda_{\max}(P)}\right)^t,
$$

which holds by a wide margin in this case. The bottom panel in Figure 4 displays instead the cumulative number of connections for each node. The blue line is the cumulative number of idle steps, i.e., time steps when no connection is requested. As we can see, the self-triggered implementation leaves the channel idle 39 times in the first 60 steps, that is, 65% of the time. The following string reports the 60 connection patterns defined by the schedule of Figure 4 (where $\cdot$ means no connection):

$$
T=\cdot\cdot\cdot\cdot\cdot\cdot1234\cdot1\cdot2\cdot1\cdot3\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot4\cdot\cdot1\cdot3\cdot12\cdot341\cdot2\cdot\cdot\cdot\cdot34\cdot\cdot\cdot\cdot\cdot1\cdot\cdot\cdot\cdot12
$$

Notice how the online-generated schedule is not periodic and does not strictly follow the sequence $1,2,3,4$, despite this being the ordering in all rotations in $\mathbf{R}$. Furthermore, notice in Figure 5 how (8) is (tightly) enforced at all times.

A. Robust centralized self-triggered scheduling

Let us now go back to the solution of Problem 2, and consider the more complex case where the nodes are subject to input noise $w_j\in W_j$, so that the feedback law in (2) becomes

$$
u(t)=k(x(t))+e(t)+w(t). \tag{20}
$$

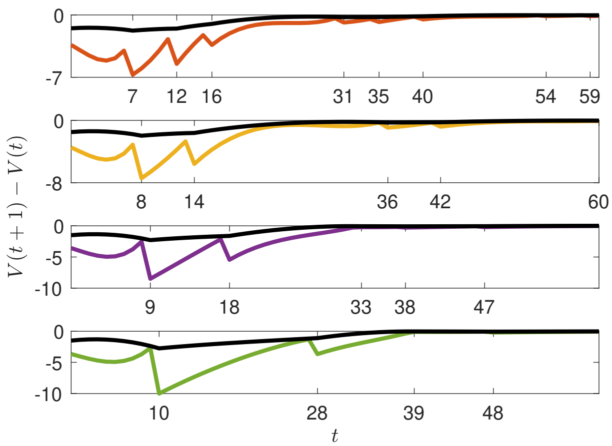

Fig. 5. In the same colour coding as for Figure 4, the value of $\boldsymbol{V(t+1)-V(t)}$ for each of the $\boldsymbol{4}$ systems. The black curve is the corresponding value of $-\boldsymbol{\sigma\alpha_3(t)}$.

The error $e(t)$ is the same as before and represents the feedback error due to the sporadic update of the state measure. The noise w is instead exogenous and may represent, for instance, a disturbance on the input or the effect on the input of a bounded measurement error. We shall approach this scenario with minimal changes to the results presented so far.

Assumption 2: The feedback law $k(x)$ makes system (20) ISS with respect to inputs e and w

This means that there exists an ISS Lyapunov function $V_i(x)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \forall x \in X$$

and

$$\begin{aligned} V(x(1,\xi,k(\xi)+e+w)) - V(\xi) \leq \\ -\alpha_3(|\xi|) + \gamma_e(|e|) + \gamma_w(|w|) \\ \forall \xi \in X, \forall e, w, \quad (21) \end{aligned}$$

where α_1, α_2, α_3, and γ_e, γ_w are $\mathcal{K}_\infty$ functions.

Assumption 3: Disturbances w in (20) are bounded, i.e., there exists a $c_w > 0$ such that

$$\gamma_w(|w|) \leq c_w \, \forall w \in W. \quad (22)$$

Theorem 10 (Problem 2 in the presence of disturbances): Under Assumption 2, the centralized self-triggered control of Theorem 9 renders the network ISS from w to x.

Proof: The online scheduling strategy of Theorem 9 ensures that

$$\begin{aligned} &\gamma_{j,e}(|k_j(x_j(t-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j))) - k(\underline{\xi}_j)|) \\ &\quad \leq (1-\sigma)\alpha_{j,3}(|x_j(t-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j))|), \forall t \geq 0. \end{aligned}$$

Using (21), this implies

$$\begin{aligned} &V_j(x_j(t-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j)))) - V_j(t-1-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j)) \\ &\quad \leq -\sigma\alpha_{j,3}(|x_j(t-1-\underline{t}_j,\underline{\xi}_j,k_j(\underline{\xi}_j))|) + \gamma_{j,w}(|w_j|). \end{aligned}$$

■

Corollary 11: Under Assumption 3 the state of the network asymptotically converges to the set

$$\{x : V(x) \leq V_w\},$$

where

$$V_w = \max_x V(x), \quad (23)$$

$$\text{s.t.} \quad \alpha_3(|x|) \leq \frac{c_w}{\sigma}. \quad (24)$$

Proof: The statement follows from the boundedness of w_i, in Assumption 3 and negative definiteness of $V(x)$ difference for $x \in \{x : \ \alpha_3(|x|) > \frac{c_w}{\sigma}\}$. ■

Example 7: Consider a set of five systems described by

$$x_j(t+1) = A_j x_j(t) + B_j\left(u_j(t) + w_j(t)\right), \quad (25)$$

where

$$A_j = \begin{bmatrix} 1 & 0.12 - 0.02 \cdot i \\ -0.05 & 1 \end{bmatrix}, \quad B_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$u_j(t) = K_j x_j(t), \quad K_j = \begin{bmatrix} -0.2 & -0.2 \end{bmatrix},$$

and the disturbance $|w_j(t)| \leq w_0$ for all $j \in \mathbb{I}_1^5$. Call $\tilde{A} := A + BK$, and assume Lyapunov functions $V := x^\top P x$, with $\tilde{A}^\top P \tilde{A} - P = -Q$ for some positive definite P and Q.

In order to find $\alpha_3(|x|)$, $\gamma_e(|e|)$, and $\gamma_w(|w|)$, one can use the results in [25][1]:

$$\alpha_3(|x|) := \frac{1}{2}\lambda_{\min}(Q)\|x(t)\|_2^2,$$

$$\gamma_e(|e|) := \left(\frac{2}{\lambda_{\min}(Q)}\|\tilde{A}^\top PB\|_2^2 + \|B^\top PB\|_2\right)\|e(t)\|_2^2,$$

$$\gamma_w(|w|) := \left(\frac{2}{\lambda_{\min}(Q)}\|\tilde{A}^\top PB\|_2^2 + \|B^\top PB\|_2\right)\|w(t)\|_2^2.$$

Considering Q as the identity matrix and using (22)-(24) and the above definitions, one can conclude

$$V_w = 2\lambda_{\max}(P)\left(2\|\tilde{A}^\top PB\|_2^2 + \|B^\top PB\|_2\right)\frac{w_0^2}{\sigma}.$$

Taking $\sigma = 0.4$ and $w_0 = 0.01$ then $V_{w,1} = V_{w,2} = 0.27$, $V_{w,3} = 0.29$, $V_{w,4} = 0.35$, $V_{w,5} = 0.57$. According to Lemma 1, we have $p_1 = p_2 = p_3 = 4$ and $p_4 = p_5 = 3$, and we can easily deduce, from Theorem 7, that the network would not be SS with the trivial connection topology. Assuming the connection patterns

$$\{C_1 := (1,2), C_2 := (2,3), C_3 := (4,5), C_4 := (5,1)\},$$

solving (11) we obtain

$$\rho^* = \left(0, \frac{1}{4}, \frac{1}{3}, \frac{1}{4}\right), \quad \eta^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\Sigma(\mathbb{I}_1^5) = \{1, 2, 5\}$ (notice that the optimization problem has multiple optimal solutions, and the selection function

[1] in [25], because of a typo, the norm of $\|B^\top PB\|$ in the γ functions is squared.

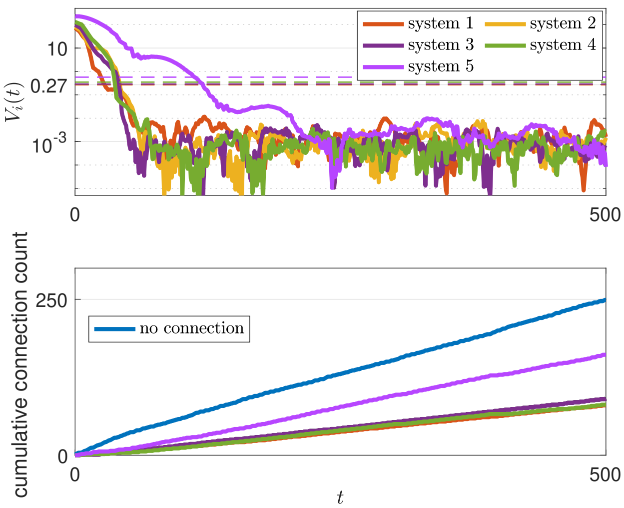

Fig. 6. The top panel shows the value of the Lyapunov functions for each of the **5** systems in Example 7, as a function of time. The bottom panel shows the cumulative number of connections of each of the systems; the blue line is the cumulative number of time instants when the communication channel is left idle.

may return multiple equivalent selections). Theorem 5 ensures the existence of a stabilizing periodic schedule for $\Sigma(\mathbb{I}_1^5)$, which in this case is the simple round robin schedule $(1, 2, 5, \ldots)$. Using Lemma 4 and the translation function, we finally obtain the stabilizing schedule for the original system: $S := (C_4, C_2, C_3, \ldots)$. A simulation of the network stabilized through the triggering strategy in Theorem 9 is shown in Figure 6. The communication channel is idle for 50% of the time, and we see in the top panel that the bounds $V_{w,i}$ overestimate the asymptotic value of the Lyapunov functions by more than one order of magnitude: our estimate is therefore far from tight.

V. Conclusions

We have analyzed the conditions for simultaneous stabilizability of a set of networked control systems both linear and nonlinear, assuming a very general formulation of the network topology using the concept of connection patterns. Quite interestingly, we have proved that a network of linear systems can be stabilized if and only if it can be stabilized by a periodic connection schedule. In terms of networked control systems theory, this means that for a network of linear systems, it is not restrictive to consider communication protocols that implement a periodic communication schedule. This does not hold, however, in the case of nonlinear systems. In this case, we have proved distinct necessary and sufficient conditions for simultaneous stabilizability. Finally, assuming one or more periodic stabilizing schedules are given, we have designed an online re-scheduling algorithm, which is effectively self-triggered, and guarantees simultaneous stability of all nodes in the network while greedily minimizing network usage, even in the presence of input disturbances. Results were illustrated by examples involving linear and nonlinear systems. The size of the networks in the examples was small to allow us to illustrate the results in a relatively compact form, but the numerical tools that we introduced scale well to much larger networks. These results expand on the existing theory, both by generalizing the possible connection topologies and by providing conditions for stabilizability that stand regardless of any choice of connection protocol.

Appendix

A. Proof of Lemma 1

In the notation of (4), we have $\alpha_3(|x|) := x^\top Qx$. Now suppose that, in (9), the input is only updated at time 0 so that $u(t) = K\xi$ for $t \in \mathbb{I}_0^{p-1}$. We have

$$x(t) = \left(A^t + \sum_{j=0}^{t-1} A^j BK\right)\xi := G(t)\xi \tag{26}$$

for $t \geq 1$, and

$$e(t) = K(\xi - x(t)) = K(I - G(t))\xi. \tag{27}$$

We can extend (26) to $t = 0$ by defining $G(0) := I$. We observe that, using (9), (26), and (27), we have

$$\begin{aligned} &V(x(t+1)) - V(x(t)) \\ &=\Big((A+BK)G(t)\xi + BK(I-G(t))\xi\Big)^\top \\ &\quad P\Big((A+BK)G(t)\xi + BK(I-G(t))\xi\Big) \\ &\quad - \xi^\top G(t)^\top PG(t)\xi \\ &\quad = \Big((AG(t)+BK)\xi\Big)^\top P\Big((AG(t)+BK)\xi\Big) \\ &\quad - \xi^\top G(t)^\top PG(t)\xi. \end{aligned}$$

Calling

$$D(t) := (AG(t)+BK)^\top P(AG(t)+BK) - G(t)^\top PG(t),$$

we can enforce $V(x(t+1)) - V(x(t)) \leq -\sigma x^\top(t)Qx(t) = -\sigma\xi^\top G^\top(t)QG(t)\xi$ by requiring that

$$D(t) \preceq -\sigma G^\top(t)QG(t). \tag{28}$$

B. Proof of Lemma 2

We need to prove that there exists at least one initial condition ξ such that (7) only holds up to some finite p. The proof is by contradiction. Assume that the pair (M^{nl}, V) is p^{nl}-step stabilizable with parameter σ^{nl} for any $p^{\text{nl}} > 0$, i.e., p^{nl} is unbounded. This implies that (8) holds for all $t \geq 0$ and for all $\xi \in \mathbb{R}^n$, and therefore that V acts as a Lyapunov function for trajectory $x(t, \xi, u(\xi))$ of M^{nl}. Therefore, $\lim_{t\to\infty} x(t, \xi, u(\xi)) = 0$. This means that $\lim_{t\to\infty} f(x(t), u(\xi)) = 0$ but also implies, by continuity of f, that $\lim_{t\to\infty} \big(f(x(t), u(\xi)) - f(0, u(\xi))\big) = 0$. Therefore,

$$f(0, u(\xi)) = 0, \; \forall\, \xi \in \mathbb{R}^n. \tag{29}$$

Now, we have by assumption that $\frac{\partial}{\partial\xi} f(\xi, u(\xi))|_{\xi=0}$ is Hurwitz, while $\frac{\partial}{\partial\xi} f(\xi, 0)|_{\xi=0}$ is unstable. Therefore, $\frac{\partial}{\partial u} f(\xi, u)|_{\xi=0,u=0} \frac{\partial}{\partial\xi} u(\xi)|_{\xi=0} \neq 0$, which implies that $\frac{\partial}{\partial\xi} f(0, u(\xi)) \neq 0$. This contradicts (29).

C. Proof of Lemma 3

Call $f^{[t]}(\xi)$ the t-time iteration of (1) obtained when keeping $u = u(0)$ constant, and set $u(0) = k(\xi)$, that is, $f^{[t]}(\xi) := x(t, \xi, k(\xi))$. We can always write the quadratic Lyapunov function $V(x)$ as $x^\top Px$ for some positive definite matrix P and, using (4) and (7), p-step stabilizability of the pair (M^{nl}, V) means that

$$\begin{aligned} &V(x(t,\xi,k(\xi))) - V(x(t-1,\xi,k(\xi))) = \\ &\quad (f^{[t]}(\xi))^\top P f^{[t]}(\xi) - (f^{[t-1]}(\xi))^\top P f^{[t-1]}(\xi) \\ &\quad\quad \leq -\sigma^{\text{nl}}\alpha_3(|f^{[t-1]}(\xi)|),\, \forall\, t \in \mathbb{I}_1^{p^{\text{nl}}}. \end{aligned} \quad (30)$$

Given that $\alpha_3(|x|) \geq 0$, $\forall\, |x| \geq 0$, this implies

$$(f^{[t]}(\xi))^\top P f^{[t]}(\xi) - (f^{[t-1]}(\xi))^\top P f^{[t-1]}(\xi) \leq 0,\, \forall\, t \in \mathbb{I}_1^{p^{\text{nl}}}.$$

Now call $J^{[t]} \in R^{n\times n}$ the Jacobian matrix of $f^{[t]}(\xi)$ at $\xi = 0$, let A, B be the matrices of the linear system M_{l}, and K the Jacobian of the feedback law $k(\xi)$ at $\xi = 0$, so that we have $J^{[t]} = \left(A^t + \sum_{j=0}^{t-1} A^j BK\right)$. Calling $H_V^{[t]}(\xi)$ the terms of the expansion of $V(x(t, \xi, k(\xi)))$ at $\xi = 0$ of cubic or higher order, we can rewrite the above inequality as

$$\begin{aligned} &\xi^\top \left(J^{[t]}\right)^\top P J^{[t]}\xi + H_V^{[t]}(\xi) \\ &\quad - \xi^\top \left(J^{[t-1]}\right)^\top P J^{[t-1]}\xi - H_V^{[t-1]}(\xi) \\ &\quad\quad \leq 0,\, \forall\, t \in \mathbb{I}_1^{p^{\text{nl}}}. \end{aligned} \quad (31)$$

This implies that $\left(J^{[t]}\right)^\top P J^{[t]} + \left(J^{[t-1]}\right)^\top P J^{[t-1]}$ is negative definite , $\forall\, t \in [1, \ldots p^{\text{nl}}]$, or else there would exist a direction ξ along which the left-hand side of (31) tends to a positive value, as $|\xi| \to 0$. Therefore there exists $c_1 > 0$ such that

$$\xi^\top \left(\left(J^{[t]}\right)^\top P J^{[t]} + \left(J^{[t-1]}\right)^\top P J^{[t-1]}\right)\xi \leq -c_1|\xi|^2,\, \forall\, t \in \mathbb{I}_1^{p^{\text{nl}}}.$$

Now consider once more (30). The fact that $V(x)$ is quadratic implies that $-\sigma^{\text{nl}}\alpha(|f^{[t-1]}(x)|)$ is lower-bounded, in a neighborhood of $\xi = 0$, by a quadratic function of $|x|$, that is,

$$-c_2|x|^2 \leq -\sigma^{\text{nl}}\alpha(|f^{[t-1]}(x)|),\, \forall\, t \in \mathbb{I}_1^{p^{\text{nl}}}$$

for some $c_2 > 0$. Clearly, there exists a sufficiently large $c_3 > 1$ such that

$$-c_1|x|^2 \leq -\frac{c_2}{c_3}|x|^2.$$

Merging the three inequalities above we obtain

$$\xi^\top \left(\left(J^{[t]}\right)^\top P J^{[t]} + \left(J^{[t-1]}\right)^\top P J^{[t-1]}\right)\xi, \leq -\frac{\sigma^{\text{nl}}}{c_3}\alpha(|f^{[t-1]}(\xi)|),\, \forall\, t \in \mathbb{I}_1^{p^{\text{nl}}}.$$

Setting $\sigma^{\text{l}} := \frac{\sigma^{\text{nl}}}{c_3}$ this implies $0 < \sigma^{\text{l}} \leq \sigma^{\text{nl}}$ and $p^{\text{l}} \geq p^{\text{nl}}$.

D. Proof of Lemma 4

The simultaneous stabilizability of the set $\Sigma(\mathbb{I}_1^q)$ with connection patterns $\hat{\mathcal{C}} := \Sigma(\mathbb{I}_1^q)$ means that a schedule $\hat{S}$ of the indices $\Sigma(\mathbb{I}_1^q)$ exists, such that index i appears at least once every p_i steps, if system $i \in \Sigma(\mathbb{I}_1^q)$ is p_i-step stabilizable. By construction of the selection function Σ and (11), for any two systems j_1 and j_2 assigned by (11) to the same connection pattern C_i (i.e., such that $\eta^*_{i,j_1} = \eta^*_{i,j_2} = 1$),

$$j_1 \in \Sigma(\mathbb{I}_1^q) \Rightarrow p_{j_1} \leq p_{j_2}, \quad (32)$$

that is, the selected (critical) nodes are the most demanding in terms of communication requirements among all those assigned to the same connection pattern. We can therefore construct a schedule S by scheduling C_i when the connection of system j such that $\eta^*_{i,j} = 1$ is scheduled in $\hat{S}$, that is, by taking $C_i := \Theta(\hat{C}_i)$. By (32), such a schedule simultaneously stabilizes all q nodes.

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